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ON LIENARD OSCILLATOR MODELS WITH A PREGIVEN SET OF LIMIT CYCLES

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Abstract—Oscillators, of the Lienard type, with an arbitrary predetermined (in number and locations) family of limit cycles are synthesized using the polynomial of the least possible degree as a nonlinear dissipative characteristic. Some new facts regarding the interplay between the set of the generating amplitudes (in the sense of quasi-linear theory) and the zeros of the nonlinear characteristic are established. On this ground a simple procedure for a concrete hardware realization of the oscillators is described. Part of the results is based on intensive numerical experiments.

1. INTRODUCTION

In recent years, in different areas of applied sciences, there have appeared models of physical or biochemical processes based on the existence of several limit cycles. In [1], two coexisting limit cycles in the phase plane correspond to different types of dynamics of cell multiplication. In [2] the authors give an example from chemical kinetics leading to a system with several limit cycles. Models of a similar mathematical nature for the description of active optical media, and models from other areas of applied physics, can be found in the works of [3–5]. Our motivation stems from the fact that while concrete devices with a single limit cycle are well known and widely described, those with multiple stable limit cycles are almost unmentioned in the literature. Perhaps such systems have never been practically synthesized, though the narrow class of Lienard oscillators with two limit cycles was discussed as a physical model of a system with metastable properties[6].

The goal of the present work is to show how to build an oscillator obeying a Lienard-type equation with a pregiven set of limit cycles; each of a specified amplitude. Simultaneously, the prescribed synthesis is optimal in the sense that it requires, as a nonlinear characteristic, a polynomial of the lowest possible degree. A broad investigation of Lienard-type oscillators with N limit cycles was given recently in [7, 8]. The mathematical ground taken in this paper, on the contrary, is restricted only to the bifurcational theory of quasi-linear systems. This type of oscillator is chosen because it is a classical topic of nonlinear oscillation theory, and because such a choice simplifies hardware realization. Needless to say, this class of oscillators is not a universally valid model for all the circumstances found in the above references.

A prospective goal, which is ultimately more interesting and important, is to obtain a picture of the interaction of many coupled oscillators; each having a set of stable limit cycles. Digital computers are likely to be incapable of adequately addressing this problem because of their inherent inefficiencies for investigation of global behavior of multidimensional systems of differential equations. Therefore the proposed electronic models can be useful in order to gain at least a qualitative understanding of the dynamical events by experimental investigations. The authors also feel that there are situations where consideration of models based on systems of limit cycles, instead of a single one, is natural and "ideologically" worthwhile. As an example of such problems in physiology, see the discussion in [9] and references therein.

2. SYSTEMS UNDER CONSIDERATION

The celebrated Van der Pol equation, allowing for one globally stable limit cycle, is widely employed as a modeling basis for numerous self-oscillating systems of quite different physical nature. As a natural generalization and extension of the Van der Pol equation to a class of systems leading to several limit cycles, we shall consider the following Lienard equation:

$$\frac{d^2x}{dt^2} + \mu W(x) \frac{dx}{dt} + x = 0, \tag{1}$$

where $W(x)$ is a real analytical function and $\mu > 0$ is a small parameter. If (1) is considered as a quasi-linear oscillator, its limit cycles should be located in the phase plane in the neighborhood of certain circles of the unperturbed linear system [$\mu = 0$, in (1)]. In order to receive more concrete results, in the following analysis $W(x)$ will be taken as a polynomial. Since the arbitrarily chosen polynomial, $W(x)$, will not necessarily result in the appearance of a specific number of periodical solutions to (1), the problem at hand is the determination of a suitable polynomial, $W(x)$, which results in the appearance of N limit cycles bifurcating from the circles,

$$x^2 + \dot{x}^2 = R_k^2, \qquad k = 1, 2, 3, \ldots, N, \tag{2}$$
$$R_1 > R_2 > \cdots > R_N > 0.$$

The circles, (2), are considered as corresponding to the periodical solutions of the unperturbed linear oscillator. Hence each limit cycle ought to be located in the neighborhood of a prescribed circle in the phase plane. (The subsequent terminology follows[10].)

Lemma 1

There exists an even polynomial, $W(x)$, of exactly degree $2N$, such that (1) allows exactly N limit cycles bifurcating from a given set of circles (2).

Proof. Determine the polynomial $B(R)$ as

$$B(R) \equiv \prod_{k=1}^{N} (R^2 - R_k^2) \equiv R^{2N} + b_{2N-2}R^{2N-2} + \cdots + b_0.$$

We are looking for the polynomial,

$$W(x) = a_{2N}x^{2N} + a_{2N-2}x^{2N-2} + \cdots + a_{2j}x^{2j} + \cdots + a_0, \tag{3}$$

such that

$$\int_0^{\pi/2} W(R\cos\phi)\sin^2\phi\, d\phi = B(R). \tag{4}$$

With the notation,

$$\delta_{2k} = \int_0^{\pi/2} (\cos^{2k}\theta) \sin^2\theta \, d\theta \qquad (k = 0, 1, \ldots, N) \tag{5}$$

we come to

$$a_{2k} = \frac{b_{2k}}{\delta_{2k}}, \quad b_{2N} = 1 \qquad (k = 0, 1, \ldots, N).$$

Q.E.D.

Remark 1

In lemma 1, only even polynomials are considered since if $W(x)$ had contained odd degrees they would not effect the radius of the circles from which the limit cycles originate. This statement follows from the fact that in the general case the bifurcating equation is (see [10], Chap. 14)

$$\int_0^{2\pi} W(R\cos(\theta)) \sin^2\theta \, d\theta = 0 \qquad (k = 0, 1, \ldots, N). \tag{6}$$

The following conjecture is based on numerous numerical experiments which will be discussed further. Conditionally, we shall call this conjecture "lemma 2."

Lemma 2

Let $W(x)$ be an even polynomial such that (1) has N simple limit cycles; each bifurcating from an appropriate cycle of the set (2). Then on each interval $[0, R_k]$, there exist at least $(N + 1 - k)$ $(k = 1, 2, \ldots, N)$ simple roots of $W(x)$. Each limit cycle with odd number is stable.

Proof. [Lemma 2 ($N = 2$)]

By virtue of the Bendixson criterion for existence of a limit cycle, it is necessary to have a sign change of the divergence of the vector field of the corresponding system for Eq. (1). Therefore $W(x)$ changes sign at least once, but then $W(x)$ should have one more real (second) root since $W(x)$ is a polynomial of second degree with respect to x^2. Both simple roots, p_1 and p_2 ($p_1 > p_2$), belong to $[0, R_1]$ due to the fact that $W(x) > 0$ for all sufficiently large x and due to the structure of Eq. (4). Since the roots of the bifurcational equation are simple, the "external" limit cycle is attractive by virtue of the standard theory of bifurcations in quasi-linear systems[10]. Q.E.D.

Remark 2

As is shown below, the separation of the roots of $B(R)$ and $W(R)$ does not necessarily take place for an arbitrary set of roots $\{R_k\}$. On the other hand, it should be noticed that the very existence of N positive simple roots of $W(x)$, by itself, does not lead automatically to the appearance of N limit cycles. (See a concrete example in Section 4.)

3. DISCUSSION OF THE CONJECTURED "LEMMA 2" AND SOME PECULIARITIES IN DISTRIBUTION OF SIGN CHANGES OF THE NONLINEARITY

Numerous intensive numerical experiments have been completed to reveal the distribution of roots of the polynomial $W(x)$ generated by the set of given amplitudes $\{R_j\}$. No contradictions to the statement of lemma 2 have been found. Moreover, in the course of these experiments, the following observation is made. For sufficiently separated neighboring limit cycles (in terms of amplitudes R_j and R_{j+1}), there is one root of $W(x)$ in the interval $[R_j, R_{j+1})$. A quantitative estimate for the term "sufficiently separated" can be observed from the Table 1, for $N = 5$.

On the other hand, one particularly interesting phenomenon is noticed when bifurcating amplitudes are sufficiently close. In the case of the Van der Pol equation, where $W(x) = x^2 - 1$, the interval $[-1, 1]$ of "negative resistance," gives rise to the only bifurcating limit cycle in the neighborhood of a circle of radius $R_1 = 2$, while $W(x)$ is positive (positive resistance) on the interval $[1, \infty)$. Therefore, intuitively the concept that if $I = [p_k, p_{k+1}]$ [see (7)] such that $W(x) < 0$, $x \in I$ and $W(p_k) = W(p_{k+1}) = 0$, then adjacent interval of "positive resistance" $[p_{k+1}, p_{k+2}]$ can have only one value R^* for the radius of the circle from which the limit cycle bifurcates, seems plausible. However, this is not the case for an oscillator with many limit cycles as demonstrated in Table 2; here, an example when one interval of negative dissipation is "followed" by several values of generating amplitudes without a sign change of $W(x)$.

Remark 3

Consider two sets of generating amplitudes $\{R_1, \ldots, R_N\}$ and $\{R_0, R_1, \ldots, R_n\}$, where $R_0 > R_1$. Let $Q = \{q_1, \ldots, q_n\}$ and $P = \{p_0, p_1, \ldots, p_N\}$ be two corresponding sets of zeros (in decreasing order) for $W(x) = W_1(x)$ and $W(x) = W_2(x)$, where $W_1(x)$ and $W_2(x)$ are polynomials (described above) of degrees $2N$ and $2N + 2$, respectively. Thus Eq. (1), with nonlinear characteristic $W_1(x)$, has N limit cycles bifurcating from circles of radii $R_1, R_2, \ldots, R_N$ and $W_2(k)$ is such that the Lienard equation has limit cycles bifurcating from the circles $R_0, R_1, \ldots, R_N$. Thorough numerical investigations confirm the following principle.

With notations introduced, it follows that

$$p_i < q_i \qquad (i = 1, \ldots, N \quad \text{and} \quad i = 0). \tag{7}$$

Loosely speaking, the values of the "previous" N roots of $W(x)$ (for $i = 0$) are decreasing as the number of succeeding members in the sequence of limit cycles increases. Table 3 demonstrates this property.

As far as the authors know, the possibility on nonseparation (see remark 2) and the

Table 1. Numerical results for $N = 5$

(a) R_j	(a) P_j	(b) R_j	(b) P_j	(c) R_j	(c) P_j
0.75	0.335	0.30	0.135	2.00	0.618
1.50	1.056	0.70	0.449	2.25	1.728
2.25	3.357	0.80	0.712	2.50	2.409
3.00	6.960	1.20	1.036	2.75	2.538
4.25	14.66	1.50	1.372	3.00	2.880

Table 2. Three successive limit cycles without change of sign of $W(x)$, for $N = 5$

P_k	R_k
1.00	0.3399
2.00	0.9499
2.07	1.354
2.11	1.443
3.00	1.644

Table 3. The root decreasing phenomenon

(a)		(b)		(c)	
R_j	P_j	R_j	P_j	R_j	P_j
1.0	0.4392	1.0	0.4325	1.0	0.4285
1.7	1.271	1.7	1.244	1.7	1.245
2.5	2.128	2.5	2.093	2.5	2.075
...	...	3.8	3.326	3.8	3.251
...	...	...	...	4.9	4.453

property expressed by (7) have never been mentioned in the literature on nonlinear oscillations.

4. COMPUTATIONAL ASPECTS OF AN OSCILLATORY DEVICE, $N = 3$

The results of Section 2 suggest the choice of $W(x)$ for simulation purposes as an even polynomial of degree $2N$,

$$W(x) \equiv a_{2N}x^{2N} + a_{2N-2}x^{2N-2} + \cdots + a_{2j}x^{2j} + \cdots + a_0, \tag{8}$$

with the coefficients defined by the set of $\{R_j\}$ prescribed amplitudes. It is important to stress that for purposes of synthesis, the representation of the polynomial, as in (8), might be inadequate from a technical point of view. Firstly, small inaccuracies in realization of coefficients can result in significant deviations from desired amplitudes of limit cycles; possibly even reducing their number. Secondly, required coefficients in (8) are likely to exceed the working range (voltage or current) of a particular electronic component(s). For example, for moderate amplitudes $\{R_k\} = \{0.5, 1, 1.5, 2, 2.5\}$, the coefficients a_{2k} for (8) are $(1, 0, -10.31, 0, 33.57, 0, -39.2, 0, 13.51, 0, -0.5768)$ which are not all within the dynamic range of most solid-state components, since scaling in such a nonlinear network is not possible. In this light, representation of $W(x)$, due to lemma 2, is taken as

$$W(x) = \prod_{k=1}^{N} (x^2 - p_k^2). \tag{9}$$

Choosing pairwise intermediate multiplications in (9) to minimize the outputs of the multipliers, one can partly alleviate the problem of not operating within the dynamic range of components. According to Section 2, $W(x)$ must have N sign changes in the interval $[0, R_1)$. Therefore the magnitudes of p_k do not exceed the value of R_1.

A circuit allowing for 2 stable limit cycles ($N = 3$) was built. Included are block and schematic diagrams, shown in Figs. 1 and 2. The data for the working parameters of the circuit are represented in Table 4. The indicated operational amplifiers have an operational range of ± 10 V. Resistors and capacitors are of the standard type, while the multipliers are transconductance integrated circuits withh a total error of $\pm 1\%$.

Table 4. Operative circuit parameters

R_j	P_j
2.00	0.747
2.40	1.953
2.80	2.574

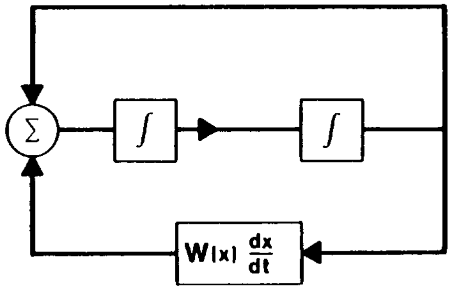

Fig. 1. Block diagram for circuit.

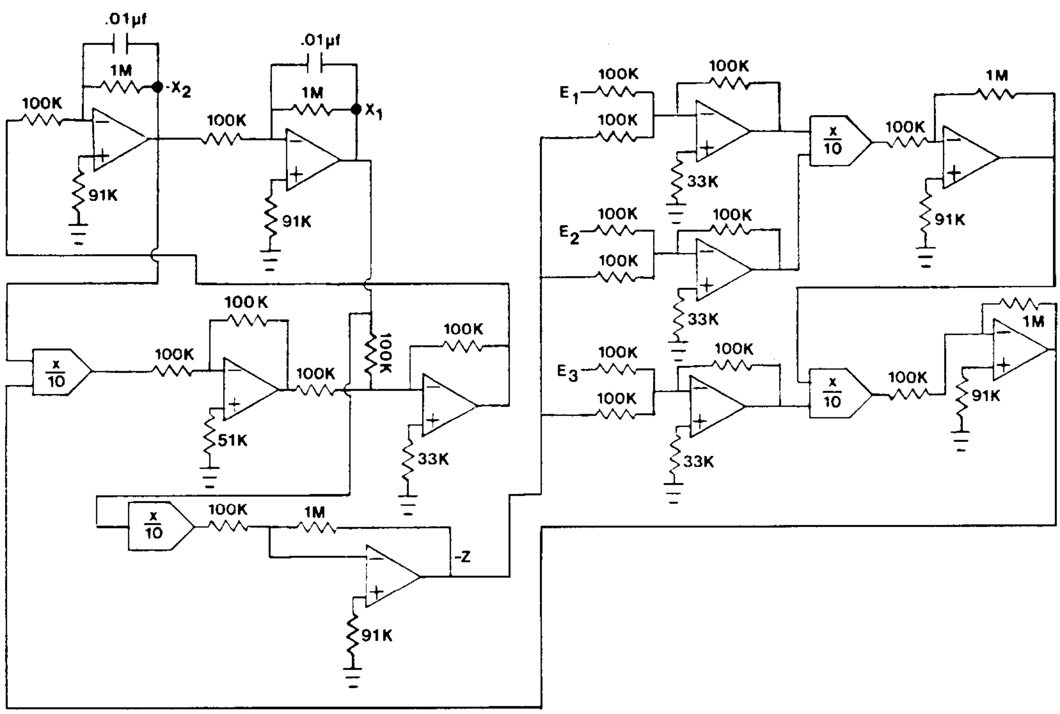

Fig. 2. Schematic diagram for circuit.

For consistency with quasi-linear theory, $\mu = 0.1$ is chosen. The time constant for the integrating portions of the circuit is chosen as 0.001 sec. No further scaling is necessary. The oscilloscope tracing of the outputs $\dot{x}(t)$ and $x(t)$ is shown in Fig. 3. Amplitudes of the limit cycles in Fig. 3 turn out to be 2.11 and 2.69 V. These results agree favorably (within 5.5%) with the theoretical values of 2.0 and 2.8, as seen in Table 4.

Figure 4 demonstrates a photograph of another oscillator excited with white noise. White-noise excitation, in this example, is only a technical means for revealing existence of multiple limit cycles in a single exposure and is shown for illustration.

As a conclusion, the synthesized systems presented are optimal in the sense that the number of sign changes of the nonlinearity, $W(x)$, gives rise to the same number of periodical regimes. The arbitrarily chosen $W(x)$ with N simple roots does not result in N limit cycles. For instance, in the case of $N = 3$, the values $p_i = 1$, $p_2 = 2$ and $p_3 = 3$, as the roots of polynomial (8), do not lead to the existence of three limit cycles.

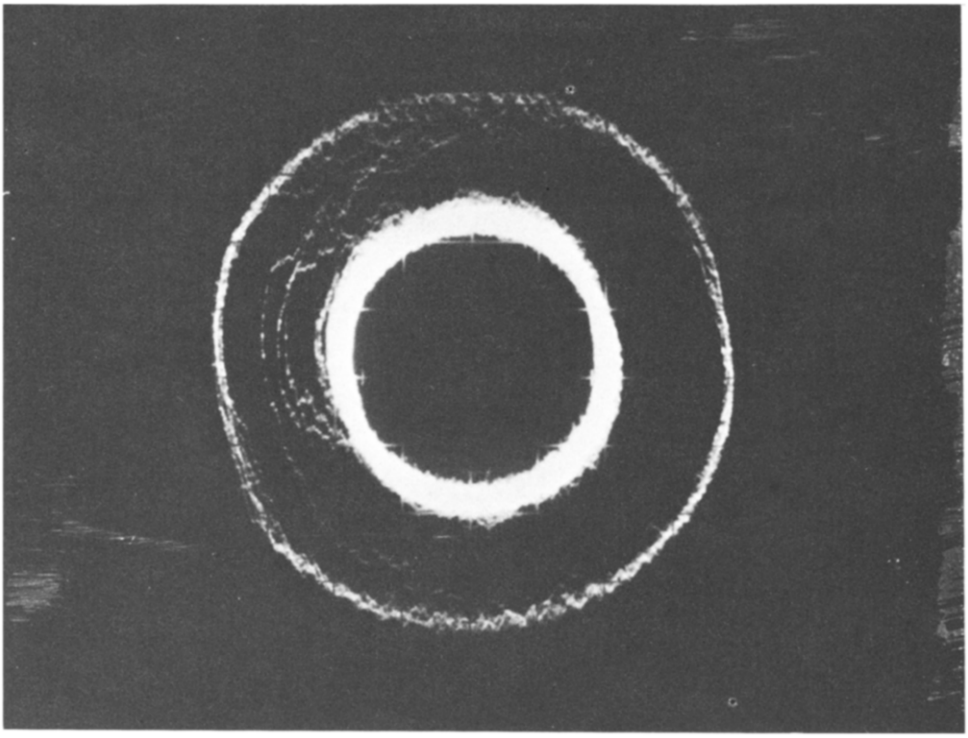

Fig. 3. Multiple exposure photograph for stable oscillations at X_1 of Fig. 2, for $N = 3$. E_1, E_2 and E_3 are specified from Table 4, as p_1^2, p_2^2 and p_3^2, respectively. Oscilloscope scale is 1 V/division.

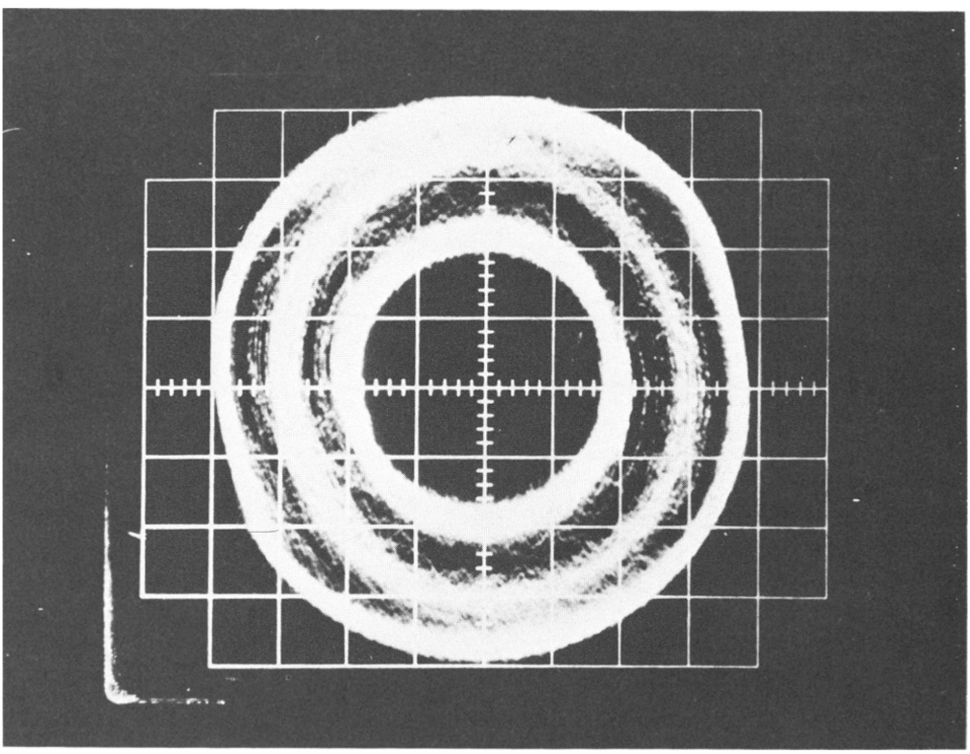

Fig. 4. Single-exposure photograph of white-noise excited oscillator with three stable limit cycles.

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